

Applications of Physics and Mathematics to Social Science

D. Stauffer* and S. Solomon

Racah Institute of Physics, Hebrew University, IL-91904 Jerusalem, Israel

*Institute for Theoretical Physics, Cologne University,
D-50923 Köln, Euroland

Article Outline

Glossary

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Glossary

Cellular Automata

Discrete variables on a discrete lattice change in discrete time steps.

Ising model

Neighbouring variables prefer to be the same but exceptions are possible. The probability for such exceptions is an exponential function of "temperature".

Percolation

Each site is randomly either occupied or empty, leading to random clusters. At the percolation threshold for the first time an infinite cluster is formed.

Universality

Certain properties are the same for a whole set of models or of real objects.

1 Definition

This article introduces into the whole section on Social Sciences, edited by A. Nowak for this Encyclopedia, concentrating on the applications of mathematics and physics. Here under "mathematics" we include also all computer simulations if they are not taken from physics, while physics applications include simulations of models which basically existed already in physics before they were applied to social simulations. Thus obviously there is no sharp border between applications from physics and from mathematics in the sense of our definition. Also social science is not defined precisely. We will include some economics as well as some linguistics, but not social insects or fish swarms, nor human epidemics or demography. Also, we mention not only this section by also the section on agent-based modelling edited by F. Castiglione as containing articles of social interest.

2 Introduction

If mathematical/physical methods are applied to social sciences, a major problem is the mutual lack of literature knowledge. Take for example the Schelling model of racial segregation in cities [1]. Sociologist don't cite the better and simpler Ising model, physicists ignored the Schelling model for decades, and sociologists also ignored better sociology work [2]. For simulations of financial markets, many econophysicists thought that they had introduced Monte Carlo and agent-based simulations to finance, not knowing of earlier work from some forward-looking Nobel laureates in economics [3, 4]. For inter-community relations, already 25 centuries ago analogies with liquids were pointed out by Empedokles in Sicily [5]. More recently, Ettore Majorana [6] around 1940 suggested to apply quantum-mechanical uncertainty to socio-economic questions. With emphasis shifted to statistical physics, sociophysics and econophysics became fashionable around the change of the millennium, but continuous lines of research by some physicists started [7] already in 1971. In the same year Journal of Mathematical Sociology started and published Schelling's model of urban segregation [1], which is a modification of the Ising magnet at zero temperature. 1982 saw the start of two other lines of research by physicists on socio-economic questions [8, 9].

Languages have been simulated on computers by decades, while the interest of physicists is more recent [10, 11], triggered mostly by a model of

language competition [12].

We do not mention chemists since at present they play no major role in this field. However, the 1921 chemistry Nobel laureate F. Soddy [13], to whom we owe the "isotope" concept, already worked on economic, social and political theories, and his finance work of the 1930's was still cited in 2007. The present authors try it the other way around: First apply physics to social sciences, and then get the Nobel prize (for literature: science fiction.)

3 Some Models and Concepts

Physicist Albert Einstein said that models should be as simple as possible, but not simpler. In this spirit we now introduce some basic physics models and concepts for readers from social sciences. They don't have to study physics for many years, the following examples give the spirit. All models are complex in the sense that the behaviour of large systems cannot be predicted from the properties of the single element.

3.1 Cellular Automata

Mathematicians denote cellular automata often as "interacting particle systems", but since many other models or methods in physics use interacting particles, we do not use this term here. A large d -dimensional lattice of L^d sites carries variables S_i ($i = 1, 2, \dots, L^d$) which can be either zero or one; more generally, they are small integers between 1 and $Q > 2$. The lattice may be square (four nearest neighbours), triangular (six nearest neighbours), or simple cubic (also six nearest neighbours, but in $d = 3$ dimensions); many other choices are also possible. Time $t = 1, 2, \dots$ increases in steps. At each time step, each $S_i(t+1)$ is calculated anew from a deterministic or probabilistic rule involving the neighbouring $S_k(t)$ of the previous time step. This way of updating is called "simultaneous" or "parallel"; one may also use sequential updating where S_i depends in the current values of S_k ; then the order of updating is important: random sequential, or regular like a typewriter.

An example is a biological infection process: Each site i becomes permanently infected, $S_i = 1$, if at least one of its nearest neighbours is already infected. Computers handle that efficiently if each computer word of, say, 32 bits stores 32 sites, and if then 32 possible infections are treated at once by bit-by-bit logical-OR operations [14].

3.2 Temperature

We know temperature T from the weather reports, but in physics it enters according to Boltzmann into the probability

$$p \propto \exp(-E/k_B T) \tag{1}$$

to observe some configuration with an energy E . Here T is the temperature measured in Kelvin (about $273 +$ the Celsius or centigrade temperature), and k_B the Boltzmann constant relating the scales of energy and temperature. For simplicity we now set $k_B = 1$, i.e. we measure temperature and energy in the same units. If g different configurations have the same energy, then $S = \ln(g)$ is called the entropy, and the probability to observe this energy is $\propto g \exp(-E/T) = \exp(-F/T)$ with the "free energy" $F = E - TS$.

In a social application we may think of peer pressure or herding: If your neighbours drink Pepsi Cola, they influence you to also drink Pepsi, even though at present you drink Coca Cola. Thus let E be the number of nearest neighbours drinking Pepsi Cola, minus the number of Coke drinking nearest neighbours. The probability for you to switch then is given by the energy difference and equal to $\exp(-2E/T)$ (or 1 if $E < 0$) in the Metropolis algorithm, or $1/(1 + \exp(2E/T))$ in the Glauber or Heat Bath algorithm. In both cases there is a tendency to decrease E . In the limit $T = 0$ one never makes a change which increases E , while for small positive T one increases E with a low but finite probability. In the opposite limit of infinite temperature, the energy becomes unimportant and all possible configurations become equally probable. Neither zero nor infinite temperature are usually realistic.

In this sense, decreasing the energy E is the most simple or most plausible choice, and the temperature measures the willingness or ability to deviate from this simplest option, e.g. to withstand peer pressure. But temperature also incorporates all those random accidents of life which influence us but are not part of the social model. For example, it may happen that there is no Pepsi Cola available even though all your neighbours drink Pepsi and you want to follow them. Investors have to make their financial choices under the influence of their clients, whose life is shaped by births, marriages, deaths, or other personal events which are not included explicitly into a financial market model. These accidents are then simulated by a finite temperature, entering the probability that one does not follow the usual rule.

The ability to withstand peer pressure and the randomness of personal lives are in principle two different things, and if one wants to include them

both one needs two different temperatures T_1 and T_2 , which do not exist in traditional physics [15].

3.3 Ising Model

In the model published by Ernst Ising in 1925, the variables S_i are not 0 or 1, but ± 1 :

$$E = - \sum_{i,k} S_i S_k - B \sum_i S_i \quad (2)$$

and for $B = 0$ this corresponds to the above Coke versus Pepsi example. The first summation runs over all neighbour pairs, the second over all sites. Thus if site i considers changing its variable, the energy change is $\pm \Delta E = 2(\sum_k S_k - B)$ and enters through $\exp(-\Delta E/T)$ into the probabilities to flip S_i ; now k runs over the neighbours of i only. (If instead of flipping one S_i one wants to exchange two different variables S_i and S_j , moving S_i into site j and S_j into site i , then one has to calculate the energy changes for both sites i and j in this "Kawasaki" kinetics.) A computer program and pictures from its application are given elsewhere in this Encyclopedia [5].

In physics, the S_i are magnetic dipole moments of the atoms, often called spins, and B is proportional to the magnetic field. Usually, physicists write an exchange constant J before the first sum, but we set $J = 1$ for simplicity here. The model was invented to describe ferromagnetism, like in the elements iron, cobalt or nickel. Later it was found to describe liquid-vapour equilibria and other phase transitions. We know that iron at room temperature is magnetic, and this corresponds to the fact that for $0 < T < T_c$ and zero field B the Ising model has the majority of its spins in one direction (either mostly $+1$ or mostly -1), while for $T > T_c$ half of the spins point in one and the other half in the opposite direction. The magnetisation $M = \sum_i S_i$, often normalised by the number L^d of spins, is therefore an order parameter. The critical temperature T_c is often named after Pierre Curie.

In one dimension, we have $T_c = 0$; in the square lattice in two dimensions we know $T_c = 2/\ln(1 + \sqrt{2})$ exactly, while on the simple cubic lattice $T_c \simeq 4.5115$ is estimated only numerically. Of course, one has generalized the model to more than nearest neighbours, to more than two states ± 1 for each spin, and to disordered lattices and networks.

3.4 Percolation

Simpler than the Ising model but less useful is percolation theory, reviewed more thoroughly in this Encyclopedia in the section edited by M. Sahimi. Each site of a large lattice is randomly occupied with probability p , empty with probability $1 - p$, and clusters are sets of occupied neighbouring sites. There is one percolation threshold p_c such that for $p < p_c$ only finite clusters exist, for $p > p_c$ also one infinite cluster, and at $p = p_c$ even several infinite clusters may co-exist, which are fractal: The number of occupied sites belonging to the infinite clusters varies at p_c as L^D where D is the fractal dimension. Here "infinite" means: spanning from one end of the sample of L^d sites to the opposite end, or: increasing in average number of sites with a positive power of L . In one dimension, again one has no phase transition ($p_c = 1$), on the square lattice $p_c \simeq 0.5927462$ and on the simple cubic lattice $p_c \simeq 0.311608$ are known only numerically, with a fractal dimension of 1, $91/48$ and $\simeq 2.5$ in one to three dimensions.

In the resulting disordered lattices, each site has from 0 to z neighbours, where z is the number of neighbours in the ordered lattice $p = 1$. If one neglects the possibility of cyclic links one finds $p_c = 1/(z - 1)$ in this Bethe lattice or Cayley tree. Near this percolation threshold the critical exponents with which several quantities diverge or vanish are the same as in the random graphs of Erdős and Rényi. But this percolation theory was published nearly two decades earlier, in 1941, by the later chemistry Nobel laureate P. Flory.

3.5 Mean Field Approximations

What is called "mean field" is called "representative agent" theory in economics, and is widespread in chemistry where the changes in the concentrations of various reacting compounds are approximated as functions of these time-dependent concentrations. A particularly simple example is Verhulst's logistic equation $dx/dt = ax(1 - x)$, known as Bass diffusion in economics. We now explain why this approximation is unreliable.

Let us return to the above Ising model of Eq.(2) and replace the S_k there by its average $\langle S_k \rangle = m = M/L^d$ which is a real number between -1 and $+1$ instead of being just -1 or $+1$; m is the normalised magnetisation. Then the total energy E is approximated as the sum over single energies E_i :

$$E = \sum_i E_i \quad E_i = (- \sum_k \langle S_k \rangle - B) S_i = -B' S_i$$

with a mean magnetic field $B' = B + \sum_k \langle S_k \rangle = B + mz$ where z again is the number of lattice neighbours. The system now behaves as if each spin S_i is in an effective field B' influenced only by the average magnetisation m and no longer directly by its neighbours S_k . The two possible orientations of S_i have the energies $\pm B'$, giving an average

$$m = \langle S_i \rangle = \tanh(B'/T) = \tanh[(B + zm)/T] \quad (3a)$$

and thus a self-consistency equation for m . Expanding the hyperbolic tangent into a Taylor series for small m and B we get

$$B = (1 - z/T)m + m^3/3 + \dots \quad (3b)$$

which gives a Curie temperature $T_c = z$, since for $T < T_c$ the magnetisation is $m = \pm[3(z/T - 1)]^{1/2} \propto (T_c - T)^{1/2}$. Similar approximations for liquid-vapour equilibria lead to the Van der Waals equation of 1872, which may be regarded as the first quantitative theory of a complex phenomenon. (m there is the difference between the liquid and the vapour density.) Nowhere in Eqs(2,3a) have we put in that there is a phase transition to ferromagnetism; it just arises from the very simple interaction energy $S_i S_k$ between neighbouring spins, and similarly the formation of raindrops emerges from the interaction between the molecules of water vapour. The water molecule is the same H_2O in the vapour, the liquid or the ice phase.

But this nice approximation contradicts the results mentioned above. For the chain, square and simple cubic lattice it predicts a $T_c = z = 2, 4$ and 6 while the correct values are $0, 2.2$, and 4.5 . Particularly in one dimension it predicts a phase transition at a positive T_c while no such transition is possible: $T_c = 0$. This was the main result of Ernst Ising's thesis in 1925. And even in three dimensions, where the difference in T_c between 4.5 and 6 is less drastic, the above square-root law for m is wrong, since m varies for T slightly below T_c roughly as $(T_c - T)^{0.32}$. Thus mean field theory, Van der Waals equation, and similar approximations averaging over many particles are at best qualitatively correct. They become exact when each particle interacts equally with all other particles.

Analogously for percolation, Flory's approximation of neglecting cyclic links and the Erdős-Rényi random graphs lead to results corresponding to mean field approximations and should not be relied upon in one, two, or three dimensions with links between nearest neighbours only.

For cellular automata a particularly drastic failure of analogous mean field approximations (differential equations) was given by Shnerb et al [16]

for a biological problem. Even simpler, many cellular automata on the square lattice lead to blinking pairs of next-nearest neighbours: at even times one site of the pair is 1 and the other is 0, while at odd times the first is 0 and the second is 1. Averaging over many sites destroys these local correlations which keep the blinking pair alive.

4 Applications

A thorough review of “sociophysics” was given recently by Castellano et al [17], a long list of reference by Carbone et al [18]. Some work of social scientists is reviewed by Davidsson and Verhagen in the section on agent-based simulations in sociology, while Troitzsch in this section reviews both social scientists and physicists. His book with Gilbert [29] is, of course, more complete. Thus we merely sketch here some the areas covered in greater detail in the other articles or in the cited literature.

4.1 Elections

A social scientist may be interested to predict the fate of one particular party or candidate in one particular election, or to explain it after this election. A physicist, accustomed to electrons, hydrogen atoms and water molecules being the same all over the world may be more interested to find which universal properties all elections have in common. Figure 1, kindly sent by Santo Fortunato, is an example. Let v be the number of votes which a candidate got, Q the number of candidates in that election, and N the total number of votes cast. Then the probability distribution $P(v, Q, N)$ for the number of votes is actually a function $f(vQ/N)$ of only one scaled variable, and that variable vQ/N is the ratio of the actual number v of votes to the average number N/Q of votes per candidate. Various countries and various elections, all using a proportional election system, gave the same curve $f(vQ/N)$ which is a parabola on this double-logarithmic plot and thus corresponds to a log-normal distribution. In Brazil, however, where the personality of a candidate play a major role, not only the party membership, the results were different. These authors [19] also present a model to explain the log-normal distribution.

Other models for opinion dynamics are reviewed elsewhere in this Encyclopedia [5].

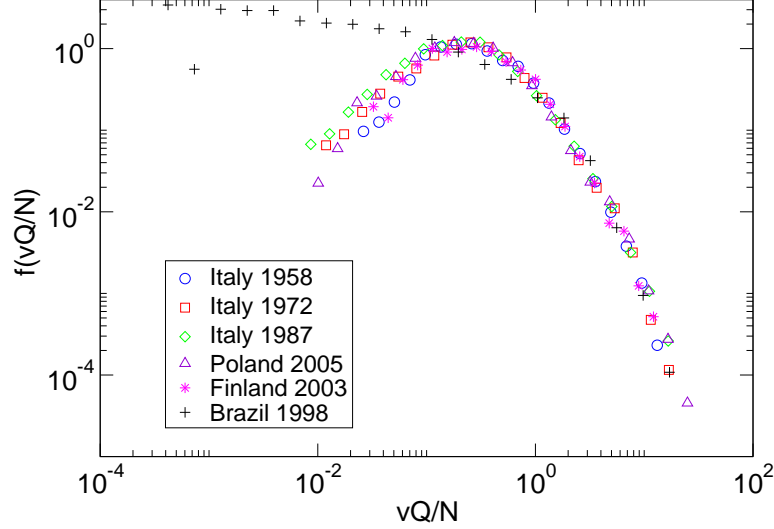


Figure 1: The vote distribution in several countries and elections is a function only of the scaled variable vQ/N . From [19].

4.2 Financial Markets

Agent-based simulation of stock markets [20] are a typical example of complex systems applications: In these models not the single agent but their (unconscious) cooperation produces the ups and downs on the stock market, the bubbles and the crashes. These models deal with the more or less random fluctuations, not with well founded market changes due to new inventions or major natural catastrophes.

Real markets give at each time interval a return r which is the relative change of the price. Typically, an index of the whole market like Dow Jones changes each trading day by about one percent. Much larger fluctuations are more rare, and the probability to have a change larger than r decays for large r as $1/r^3$: Fat tails. The sign of the change is barely predictable, but its absolute value is: Volatility clustering. Thus in calm times when $|r|$ was small, tomorrow's $|r|$ probably is also small, whereas for turbulent times with high $|r|$ in the past one should also expect a large $|r|$ tomorrow. The daily weather behaves similarly: presumably tomorrow will be like today. Perhaps even multifractality exists in real markets, similar to hydrodynamic

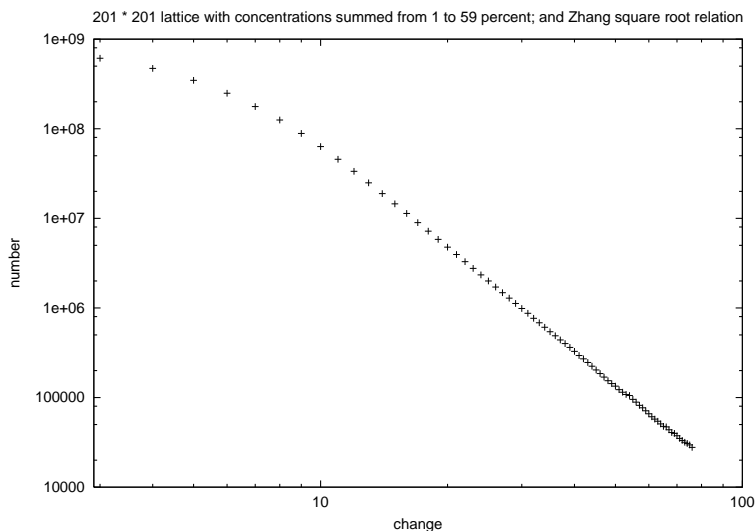


Figure 2: Simulated return distribution in the Cont-Bouchaud percolation model of stock markets [21]. The asymptotic slope to the right is about -2.9 .

turbulence.

A simple model, going back to Bachelier more than a century ago, would throw a coin to determine whether the market tomorrow will go up or down. This simple random-walk or diffusion model was shown by Mandelbrot in the 1960's not to describe a real market; it lacks fat tails and volatility clustering but may be good for monthly changes. Many better agent-based models have been invented during the last decade and reproduce these real properties, Fig. 2; the Cont-Bouchaud model is based on the above percolation theory [21], while the Minority Game tells you it is better not to be with the big crowd [22].

4.3 Languages

The versatility of human languages distinguishes us from the simpler communication systems of other living beings. With computers or mathematically exact solutions [23] models have been studied for the learning of a language by children or for the evolution of human languages out of simpler forms.

Closer to simulations in biology with the Darwinian selection of the fittest are the models of competition between various languages of adult humans:

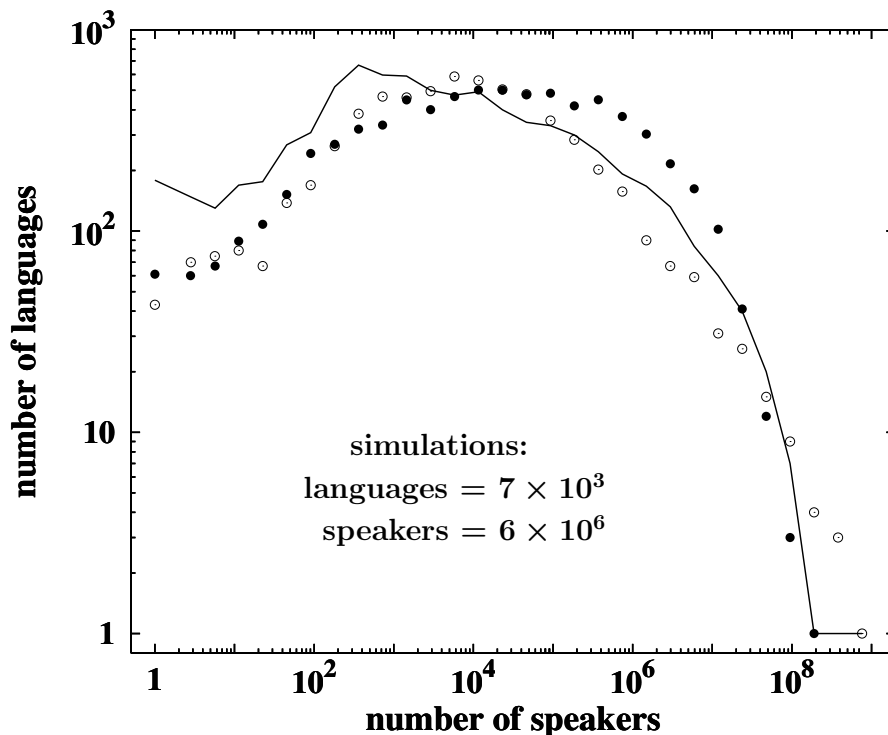


Figure 3: Simulated size distributions for human languages (full circles, and line), compared with reality (open circles). From [25].

Will the Welsh language survive against English in Great Britain? Similar to Lotka-Volterra equations for prey and predator in biology, some nonlinear differential equations [12] seem to describe the extinction of the weaker language. Better statistics are available for the size distribution of languages, where "size" is the number of people speaking this language. Here one model of de Oliveira et al found good agreement with reality, Fig.3; other models [24] were less successful, inspite of many simulations from physicists.

5 Future Directions

The future should see more work in what we have shown here through our three figures: Searching for universal properties, or the lack of them, in the multitudes of models and in reality. Biology became a real science when the

various living beings were classified into horses, mammals, vertebrates etc. Within each such taxonomic set all animals have certain things in common, which animals in other taxonomic sets do not share. This check for universality is different from improving our ability to ride horses. Thus making money on the stock market, or explaining the crash of 1987, is nice, but investigating the exponents of the fat tails, Fig.2, of all markets may give us more insight into what drives a market and what differences exist between different markets. Winning one particular election and predicting the winner is important, but universal scaling properties as in Fig.1 may help us to understand democracy better. Preventing the extinction of French language in Canada is important for the people there, but explaining the overall statistics of languages in Fig.3 is relevant globally.

It is in these general aspects where the methods of mathematics and physics seem to be most fruitful. One specific problem is better solved by the local people who know that problem best, not by general simplified models. A useful future approach for interacting agents would be their realisation by neural network models [26].

References

[1] **Primary literature:**

- Schelling TC (1971) J. Math. Sociol. 1: 143
- [2] Dethlefsen E, Moody C (July 1982) Byte 7: 178; Jones FL (1985) Aust. NZ J. Sociol. 21: 431
- [3] Stigler GJ (1964) J. Business 37: 117
- [4] Kim GW, Markowitz HM (Fall 1989) J. Portfolio Management 16: 45
- [5] Stauffer D (2008) Opinion Dynamics and Sociophysics. Draft for this section in this Encyclopedia. arXiv:0705.0891
- [6] Mantegna RN (2005) Presentation of the English translation of Ettore Majorana's paper: "The value of statistical laws in physics and social sciences". Quantitative Finance 5: 133

- [7] Weidlich W (2000) *Sociodynamics; A Systematic Approach to Mathematical Modelling in the Social Sciences* Harwood Academic Publishers; 2006 reprint: Dover, Mineola (New York)
- [8] Galam S (2008) Int. J. Mod. Phys. C 19: issue 3; Galam S, Gefen Y, Shapir Y (1982) J. Math. Sociol. 9: 1
- [9] Roehner B, Wiese KE (1982) Environment and Planning A 14: 1449
- [10] Gomes MAF, Vasconcelos GL, Tang IJ, Tang IR (1999) Physica A 271: 489
- [11] Zanette D (2001) Adv. Complex Syst. 4: 281
- [12] Abrams MH, Strogatz SH (2003) Nature 424: 900
- [13] http://nobelprize.org/nobel_prizes/chemistry/laureates/1921/soddy-bio.html
- [14] Stauffer Di (1991) J. Phys. A 24: 909
- [15] Ódor G (2008) Int. J. Mod. Phys. C 19: issue 3
- [16] Shnerb NM, Louzoun Y, Bettelheim E, Solomon S (2000) Proc. Natl. Acad. Sci. USA 97: 10322
- [17] Castellano C, Fortunato S, Loreto V (2007) arXiv:0710.3256 submitted to Reviews of Modern Physics.
- [18] Carbone A, Kaniadakis G, Scarfone AM (2007) Eur. Phys. J. B 57: 121
- [19] Fortunato S, Castellano C (2007) Phys. Rev. Lett. 99: 138701.
- [20] Levy M, Levy H, Solomon S (2000) *Microscopic Simulation of Financial Markets*, Academic Press, New York; Samanidou E, Zschichang R., Stauffer D, Lux T (2007) Rep. Progr. Phys. 70: 404, Lux T (2007) "Stochastic behavioral asset pricing models and the stylized facts", draft for *Handbook of Finance*, Hens T, Schenk-Hoppé K
- [21] Stauffer D (2001) Adv. Complex Syst.4: 19
- [22] Challet D, Marsili M, Y.-C. Zhang YC (2004) *Minority Games*, Oxford University Press, Oxford.

- [23] Komarova NL (2004) J. Theor. Biology 230, 227
- [24] Schulze S, Stauffer D, Wichmann S (2007) Comm. Comput. Phys. 3: 271
- [25] de Oliveira PMC, Stauffer D, Wichmann S, Moss de Oliveira S (2007) arXiv:0709.0868
- [26] Klüver J (2008) part VI.2 in "Social-cognitive complexity", this section in this Encyclopedia; Wischmann S, Hulse M, Knabe JF, Pasemann F (2006) Adaptive Behavior 14: 117; Stauffer D (2007) arXiv:0712.4364
- [27] **Books and Reviews:**
 - Stauffer D, Moss de Oliveira S, de Oliveira PMC, Sá Martins JS (2006) *Biology, Sociology, Geology by Computational Physicists*, Elsevier, Amsterdam
- [28] Billari FC, Fent T, Prskawetz A, Scheffran J (2006) *Agent-based computational modelling*, Physica-Verlag, Heidelberg
- [29] Gilbert N, Troitzsch KG (2005) *Simulation for the Social Scientist*, 2nd edition, Open University Press, Maidenhead and New York